

# Tracking Control of Fully-actuated Mechanical port-Hamiltonian Systems using Sliding Manifolds and Contraction

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**Abstract:** In this paper, we propose a novel trajectory tracking controller for fully-actuated mechanical port-Hamiltonian (pH) systems, which is based on recent advances in contraction-based control theory. Our proposed controller renders a desired sliding manifold (where the reference trajectory lies) attractive by making the corresponding error system partially contracting. Finally, we present numerical simulation results where a SCARA robot is commanded by our proposed tracking control law.

**Keywords:** Trajectory tracking control, port-Hamiltonian systems, sliding manifold, differential Lyapunov theory, partial contraction

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## 1. INTRODUCTION

The control of electro-mechanical (EM) systems is a well-studied problem in control theory literature. Using Euler-Lagrange (EL) formalism for describing the dynamics of EM systems, many control design tools have been proposed and studied to solve the stabilization/set-point regulation problem. Recent works include the passivity-based control methods which are expounded in Ortega et al. (2013) and references therein. However, for motion control/output regulation problem (which includes trajectory tracking and path-following problems), the use of EL formalism in the control design is relatively recent and the problem is solved based on passivity/dissipativity theory for nonlinear systems (Willems (1972)). We refer interested reader on the early work of tracking control for EL systems in Slotine and Li (1987) and recent works in Kelly et al. (2006); Jayawardhana and Weiss (2008).

As an alternative to the EL formalism for describing EM systems, port-Hamiltonian (pH) framework has been proposed and studied (see also the pioneering work in van der Schaft and Maschke (1995) and the recent exposition in van der Schaft and Jeltsema (2014)), which has a nice (Dirac) structure, provides port-based modeling and has physical energy interpretation. For the latter part, the energy function can directly be used to show the dissipativity and stability property of the systems. The port-based modeling of pH systems is modular in the sense that we

can interconnect pH systems through their external ports with a power-preserving interconnection that still preserve the pH structure of the interconnected system.

Using the pH framework, a number of control design tools have been proposed and implemented for the past two decades. For solving the stabilization and set-point regulation problem of pH systems, we can apply, for instance, the standard proportional-integral (PI) control (Jayawardhana et al. (2007)), Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) approach (Ortega et al. (2002)) or Control by Interconnection (CbI) method (Ortega et al. (2008)) (among many others).

However, for motion control problem, where the reference signal can be time-varying, it is not straightforward to design control laws for such pH systems that still provides an insightful energy interpretation of the closed-loop system. For example, it is not trivial to obtain an incremental passive system<sup>1</sup> via a controller interconnected with the pH system. One major difficulty is that the external reference signals can induce both the closed-loop system and total energy function to be time-varying. In this case, the closed-loop system may not be dissipative, or if it is a time-varying dissipative system, the usual La-Salle invariance principle argument can no longer be invoked for analyzing the asymptotic behavior.

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<sup>1</sup> This concept generalizes the usual notion of passivity and is suitable for output regulation problem of non-constant signals, see Jayawardhana (2006), Pavlov and Marconi (2006)

In order to overcome the loss of passivity in the trajectory tracking control of pH systems, a pH structure preserving error system was introduced in Fujimoto et al. (2003) which is based on generalized canonical transformations. In Fujimoto et al. (2003), necessary and sufficient conditions for passivity preserving are given. Once in the new canonical coordinates, the pH error system can be stabilized with standard passivity-based control methods.

In Dirks and Scherpen (2010), the previous approach is extended to an adaptive control one. In Romero et al. (2015a) and similarly in Romero et al. (2015b), a generalized canonical transformation is used to obtain a particular pH system which is *partially linear* in the momentum with constant inertia matrix. The control scheme is then proposed to give a pH structure for the closed-loop error system. Although solving partial differential equations that correspond to the existence of such transformation is not trivial, some characterizations of this canonical transformation is presented in Venkatraman et al. (2010) for a specific class of systems. In Yaghmaei and Yazdanpanah (2015), the so-called *timed-IDA-PBC* was introduced, where the standard IDA-PBC method for stabilization is adapted in such a way that it can incorporate tracking problem through a modification in the IDA-PBC matching equations; albeit it may easily lead to a non-tractable problem in solving a set of complex PDE. Finally, in a recent paper Zada and Belda (2016), a trajectory tracking control for standard pH systems without dissipation is proposed, using a similar change of coordinates as in Slotine and Li (1987) where the *Coriolis* term is defined explicitly in the Hamiltonian domain. This approach is motivated by the work of Arimoto (1996).

As an alternative to the use of passivity-based control method for solving motion control problems, we propose in this paper a contraction-based control method for fully-actuated pH systems. The main idea is to combine the recent result in transverse exponential stability (Andrieu et al. (2016)), where an invariant manifold is made attractive through a contraction-based control law, and the sliding-manifold approach (Ghorbel and Spong (2000)), where we ensure that on the invariant manifold, the trajectory converges to the desired reference trajectory via an additional control law. There already exist different approaches to get attractivity of the sliding manifold  $\mathcal{S}$ , such as, sliding mode control (Utkin (2013)), singular perturbations techniques (Ghorbel and Spong (2000)), passivity (Slotine and Li (1987)), Immersion & Invariance (I&I) approach (Wang et al. (2016)) and reduction methods (El-Hawary and Maggioro, 2013).

Generally speaking, we first construct an error system using backstepping method such that the closed-loop error system (in the sense of Fujimoto et al. (2003)) has a time-varying Hamiltonian-like structure, but not necessarily with a constant inertia matrix as in (Romero et al. (2015b)). In order to show the convergence of the system's trajectory to the time-varying reference trajectory, the state space is extended with the incorporation of a *virtual system* where the latter system admits both the system's trajectory, as well as, the reference trajectory as its solution. By using the transverse exponential stability results as in (Andrieu et al. (2016)) or the *partial contraction* as in (Wang and Slotine (2005)), the contraction of the

virtual system in the extended system implies that state trajectory converges exponentially to the desired one.

## 2. PRELIMINARIES

### 2.1 Control design using sliding manifolds

Let  $\mathcal{X}$  be the state-space with tangent bundle  $T\mathcal{X}$  of a nonlinear system given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{G}(\mathbf{x}, t)\mathbf{u} \quad (1)$$

where  $\mathbf{G}(\mathbf{x}, t) = [\mathbf{g}_1(\mathbf{x}, t), \dots, \mathbf{g}_n(\mathbf{x}, t)]$  has full rank and  $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^N$  the solution to (1),  $\mathbf{f}, \mathbf{g}_k$  are smooth vector fields on  $\mathcal{X} \times \mathbb{R}_{\geq 0}$  for  $k \in \{1, \dots, n\}$  and  $\mathbf{u} = [u_1, \dots, u_n]^\top$  the control input. Recall the definitions of invariant and sliding manifolds.

*Definition 1.* (Ghorbel and Spong (2000)). The set  $\mathcal{S} \subset \mathcal{X}$  is said to be an *invariant manifold* for system (1) if whenever  $\mathbf{x}(t_0) \in \mathcal{S}$ , implies that  $\mathbf{x}(t) \in \mathcal{S}$ , for all  $t > t_0$ .

*Definition 2.* (Sira-Ramrez (2015)). A *sliding manifold* for system (1) is a subset of the state space, which is the intersection of  $n$  smooth  $(N - 1)$ -dimensional manifolds,

$$\Omega(t) = \{\mathbf{x} \in \mathcal{X} : \boldsymbol{\sigma}(\mathbf{x}, t) = 0\} \quad (2)$$

where  $\boldsymbol{\sigma}(\mathbf{x}, t) = [\sigma_1(\mathbf{x}, t), \dots, \sigma_n(\mathbf{x}, t)]^\top$  is the *sliding variable* with  $\sigma_i$  a smooth function  $\sigma_i : \mathcal{X} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ .

It is assumed that  $\Omega(t)$  is locally an  $(N - n)$ -dimensional, sub-manifold of  $\mathcal{X}$ . The smooth control vector  $\mathbf{u}_{eq}$ , known as the *equivalent control*, renders the manifold  $\Omega$  to an invariant manifold  $\mathcal{S}$  of (1) (Utkin (2013)). If  $\text{rank}\{L_{\mathbf{G}}\boldsymbol{\sigma}\} = n$ , the equivalent control is the well defined solution to the following invariance conditions

$$\boldsymbol{\sigma}(\mathbf{x}, t) = 0, \quad \dot{\boldsymbol{\sigma}}(\mathbf{x}, t) = 0, \quad (3)$$

uniformly in  $t$ . The dynamical system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{G}(\mathbf{x}, t)\mathbf{u}_{eq}(\mathbf{x}, t)$  is said to describe the *ideal sliding motion*.

Using sliding manifolds in control design has as goal, designing a suitable control scheme  $\mathbf{u} = \mathbf{u}_{eq} + \mathbf{u}_{at}$ , such that  $\mathbf{u}_{eq}$  renders  $\Omega(t)$  to an invariant sliding manifold, under invariance conditions (3), and  $\mathbf{u}_{at}$  makes to the invariant manifold attractive.

### 2.2 Contraction analysis and differential Lyapunov theory

System (1) in closed-loop with  $\mathbf{u}$ , denoted by

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t), \quad (4)$$

is called *contracting*, if initial any pair of solution  $\mathbf{x}_1$  and  $\mathbf{x}_2$  converges to each other, with respect to a *distance*. In this paper, for contraction analysis, we adopt the approach given in Forni and Sepulchre (2014).

The *prolonged system* (Crouch and van der Schaft (1987)) of (4) corresponds to the original system *together* with its *variational system*, that is the system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t) \\ \delta\dot{\mathbf{x}} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{x}, t)\delta\mathbf{x} \end{cases} \quad (5)$$

with  $(\mathbf{x}, \delta\mathbf{x}, t) \in T\mathcal{X} \times \mathbb{R}_{\geq 0}$ . A *differential Lyapunov function*  $V : T\mathcal{X} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  satisfies the bounds

$$c_1 \|\delta\mathbf{x}\|_{\mathbf{x}}^p \leq V(\mathbf{x}, \delta\mathbf{x}, t) \leq c_2 \|\delta\mathbf{x}\|_{\mathbf{x}}^p, \quad (6)$$

where  $c_1, c_2 \in \mathbb{R}_{>0}$ ,  $p$  is some positive integer and  $\|\cdot\|_{\mathbf{x}}$  is a Finsler structure. The role of (6) is to measure the

distance of any tangent vector  $\delta \mathbf{x}$  from  $\mathbf{0}$ . Thus, (6) can be understood as a classical Lyapunov function for the linearized dynamics with respect to the origin in  $T\mathcal{X}$ .

*Theorem 1.* Consider the prolonged system (5), a connected and forward invariant set  $\mathcal{D}$ , and a strictly increasing function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ . Let  $V$  be a differential Lyapunov function satisfying

$$\dot{V}(\mathbf{x}, \delta \mathbf{x}, t) \leq -\alpha(V(\mathbf{x}, \delta \mathbf{x}, t)) \quad (7)$$

for each  $(\mathbf{x}, \delta \mathbf{x}) \in T\mathcal{X}$  and uniformly in  $t \in \mathbb{R}_{\geq 0}$ . Then, (4) contracts  $V$  in  $\mathcal{D}$ .  $V$  is called the *contraction measure*, and  $\mathcal{D}$  the *contraction region*.

*Remark 1.* Contraction of (4) is guaranteed by (6) and (7), with respect to the distance induced by the Finsler measure  $\|\cdot\|_{\mathbf{x}}$ , through integration. As direct consequence (Forni and Sepulchre (2014)), system (4) is incrementally

- *stable* on  $\mathcal{D}$  if  $\alpha(s) = 0$  for each  $s \geq 0$ ;
- *asymptotically stable* on  $\mathcal{D}$  if  $\alpha$  is a strictly increasing;
- *exponentially stable* on  $\mathcal{D}$  if  $\alpha(s) = \beta s, \forall s > 0$ .

*Remark 2.* By taking as differential Lyapunov function to  $V(\mathbf{x}, \delta \mathbf{x}) = \frac{1}{2} \delta \mathbf{x}^\top \mathbf{\Pi}(\mathbf{x}, t) \delta \mathbf{x}$ , with  $\mathbf{\Pi}(\mathbf{x}, t)$  a smooth Riemannian metric and uniform in  $t$ , expression (7) results in the so-called *generalized contraction analysis* in Lohmiller and Slotine (1998), i.e.,

$$\frac{\partial \mathbf{\Pi}}{\partial \mathbf{x}} \mathbf{F}(\mathbf{x}, t) + \frac{\partial \mathbf{F}^\top}{\partial \mathbf{x}} \mathbf{\Pi}(\mathbf{x}, t) + \mathbf{\Pi}(\mathbf{x}, t) \frac{\partial \mathbf{F}}{\partial \mathbf{x}} < 2\beta \mathbf{\Pi}. \quad (8)$$

If the interest is on convergence to a specific trajectory, rather than convergence between any two arbitrary trajectories. The concept introduced by Wang and Slotine, 2005, with the name of *partial contraction*, gives a solution

*Theorem 2.* Assume (4) is written as the actual system

$$\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}, \mathbf{x}, t), \quad (9)$$

Consider a virtual system

$$\dot{\mathbf{x}}_v = \mathbf{h}(\mathbf{x}_v, \mathbf{x}, t). \quad (10)$$

such that both  $\mathbf{x}_v = \mathbf{x}$  and a smooth trajectory  $\mathbf{x}_v = \mathbf{x}_d(t)$  are particular solutions to (10). If the virtual system is contracting, uniformly in  $\mathbf{x}$  and  $t$ , then  $\mathbf{x}$  (asymptotically/exponentially) converges to  $\mathbf{x}_d(t)$ . System (9) is said to be partially contracting.

*Remark 3.* From a control design perspective, we want the control system (1) to track a given desired trajectory  $\mathbf{x}_d$ . To that end, in Jouffroy and Fossen (2010), an adaptation of Theorem 2 was presented as follows. Suppose the control system (1) is rewritten as the actual system

$$\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}, \mathbf{x}, \mathbf{x}_d, \mathbf{u}, t), \quad (11)$$

and assume that the controller equation  $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{x}_d, \dot{\mathbf{x}}_d)$  can be expressed in implicit form by

$$\dot{\mathbf{x}}_d = \mathbf{h}(\mathbf{x}_d, \mathbf{x}, \mathbf{x}_d, \mathbf{u}, t), \quad (12)$$

where  $\mathbf{x}_d$  is a desired trajectory. Consider now as virtual system to

$$\dot{\mathbf{x}}_v = \mathbf{h}(\mathbf{x}_v, \mathbf{x}, \mathbf{x}_d, \mathbf{u}, t). \quad (13)$$

If system (13) is contracting uniformly in  $\mathbf{x}, \mathbf{x}_d$  and  $t$ , then conclusion of Theorem 2 holds.

### 2.3 Mechanical port-Hamiltonian systems

Consider the *input-state port-Hamiltonian* (van der Schaft and Jeltsema (2014)) representation of a fully-actuated mechanical system of the form

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n & \mathbf{I}_n \\ -\mathbf{I}_n & -\mathbf{D}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \mathbf{q}}(\mathbf{q}, \mathbf{p}) \\ \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}, \mathbf{p}) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_n \\ \mathbf{G}(\mathbf{q}) \end{bmatrix} \mathbf{u}, \quad (14)$$

where  $\mathbf{x} = [\mathbf{q}, \mathbf{p}]^\top \in T^*\mathcal{Q} = \mathcal{X}$ , and the momentum vector defined by  $\mathbf{p} = \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ , with  $\dot{\mathbf{q}}$  the generalized velocity vector and  $\mathbf{M}(\mathbf{q}) = \mathbf{M}^\top(\mathbf{q}) > \mathbf{0}_n$  the inertia matrix,  $\mathbf{D}(\mathbf{q}) = \mathbf{D}^\top(\mathbf{q}) \geq \mathbf{0}_n$  is the damping matrix, square matrices  $\mathbf{I}_n, \mathbf{0}_n$  have dimension  $n = \dim \mathcal{Q}$ ,  $\mathbf{G}(\mathbf{q})$  is the full-rank input matrix,  $\mathbf{u} \in \mathbb{R}^n$  the control input and the Hamiltonian function is given by the total energy

$$H(\mathbf{x}) = \frac{1}{2} \mathbf{p}^\top \mathbf{M}^{-1}(\mathbf{q}) \mathbf{p} + V(\mathbf{q}), \quad (15)$$

with  $V(\mathbf{q})$  the potential energy. In Arimoto (1996) it was proven that the matrix  $\mathbf{S}(\mathbf{q}, \dot{\mathbf{q}})$  (which is skew-symmetric, homogeneous and linear in  $\dot{\mathbf{q}}$ ), defined by

$$S_{ij}(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2} \left[ \sum_{k=1}^n \dot{q}_k \left( \frac{\partial M_{ik}}{\partial q_j}(\mathbf{q}) - \frac{\partial M_{jk}}{\partial q_i}(\mathbf{q}) \right) \right], \quad (16)$$

where  $S_{ij} = -S_{ji}$ , fulfills the property

$$\left[ \mathbf{S}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{1}{2} \dot{\mathbf{M}}(\mathbf{q}) \right] \dot{\mathbf{q}} = -\frac{\partial}{\partial \mathbf{q}} \left[ \frac{1}{2} \dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} \right]. \quad (17)$$

Such a property is for the Euler-Lagrange realization of mechanical systems with state variables  $(\mathbf{q}, \dot{\mathbf{q}})$ . However, as was shown in Zada and Belda (2016), by applying the *Legendre transformation*, property (17) can be expressed in states  $(\mathbf{q}, \mathbf{p})$  of the Hamiltonian realization (14) as

$$\left[ \mathbf{S}(\mathbf{q}, \mathbf{p}) - \frac{1}{2} \dot{\mathbf{M}}(\mathbf{q}) \right] \mathbf{M}^{-1}(\mathbf{q}) \mathbf{p} = \frac{\partial}{\partial \mathbf{q}} \left[ \frac{1}{2} \mathbf{p}^\top \mathbf{M}^{-1}(\mathbf{q}) \mathbf{p} \right]. \quad (18)$$

Using (18), system (14) can be rewritten as

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n & \mathbf{I}_n \\ -\mathbf{I}_n & -\mathbf{E}(\mathbf{q}, \mathbf{p}) \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial \mathbf{q}}(\mathbf{q}) \\ \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}, \mathbf{p}) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}(\mathbf{q}) \end{bmatrix} \mathbf{u} \quad (19)$$

with  $\mathbf{E}(\mathbf{q}, \mathbf{p}) := \mathbf{S}(\mathbf{q}, \mathbf{p}) - \frac{1}{2} \dot{\mathbf{M}}(\mathbf{q}) + \mathbf{D}(\mathbf{q})$ . Notice that system (19) preserves passivity, with storage function (15).

## 3. TRAJECTORY TRACKING CONTROLLER

*Control objective:* Design a control scheme for system (14) such that  $\mathbf{x}$  converges to a given twice differentiable desired trajectory  $\mathbf{x}_d(t) \in T^*\mathcal{Q}$ .

To solve the control problem, it is necessary to construct a suitable error system for (14), as in Fujimoto et al. (2003). Consider a twice differentiable desired trajectory  $\mathbf{x}_d = [\mathbf{q}_d(t), \mathbf{p}_d(t)]^\top$ , with  $\mathbf{p}_d(t) = \mathbf{M}(\mathbf{q}_d(t))\dot{\mathbf{q}}_d(t)$  and a change of coordinates

$$\tilde{\mathbf{x}} := \begin{bmatrix} \tilde{\mathbf{q}} \\ \tilde{\boldsymbol{\sigma}} \end{bmatrix} = \begin{bmatrix} \mathbf{q} - \mathbf{q}_d(t) \\ \mathbf{p} - \mathbf{p}_r(t) \end{bmatrix} \quad (20)$$

where  $\mathbf{p}_r$  is an auxiliary momenta reference to be defined. The dynamics of the first component of (20) is

$$\dot{\tilde{\mathbf{q}}} = \mathbf{M}^{-1}(\tilde{\mathbf{q}} + \mathbf{q}_d) \mathbf{p} - \mathbf{M}^{-1}(\mathbf{q}_d) \mathbf{p}_d. \quad (21)$$

Like in *backstepping*, assume  $\mathbf{p} = \boldsymbol{\sigma} + \mathbf{p}_r$  is a control input to (21), with  $\boldsymbol{\sigma}$  as new state and  $\mathbf{p}_r$  as a stabilizing term. After substitution of  $\mathbf{p}$  and defining to  $\mathbf{p}_r$  as

$$\mathbf{p}_r = \mathbf{p}_{d\sigma} - \Lambda \tilde{\mathbf{q}} \quad (22)$$

for  $\mathbf{p}_{d\sigma} = \mathbf{M}(\tilde{\mathbf{q}} + \mathbf{q}_d)\dot{\mathbf{q}}_d$  and  $-\mathbf{\Lambda}$  is a Hurwitz matrix, it results in the position error dynamics

$$\dot{\tilde{\mathbf{q}}} = \mathbf{M}^{-1}(\tilde{\mathbf{q}} + \mathbf{q}_d)(\boldsymbol{\sigma} - \mathbf{\Lambda}\tilde{\mathbf{q}}), \quad (23)$$

with  $\boldsymbol{\sigma}$  as input. When  $\boldsymbol{\sigma} = \mathbf{0}$  in (23), the origin  $\tilde{\mathbf{q}} = \mathbf{0}$  is asymptotically stable, since  $-\mathbf{M}^{-1}(\tilde{\mathbf{q}} + \mathbf{q}_d)\mathbf{\Lambda}$  is a Hurwitz matrix. Above implies  $\mathbf{q} \rightarrow \mathbf{q}_d$  as  $t \rightarrow \infty$ . Simultaneously, from (22),  $\mathbf{p}_r \rightarrow \mathbf{p}_d$  as  $t \rightarrow \infty$ .

The dynamics of  $\boldsymbol{\sigma}$  is simply  $\dot{\boldsymbol{\sigma}} = \dot{\mathbf{p}} - \dot{\mathbf{p}}_r$  evaluated in the change of variables (20). Then, an error system for (14) is

$$\begin{aligned} \dot{\tilde{\mathbf{q}}} &= \mathbf{M}^{-1}(\tilde{\mathbf{q}} + \mathbf{q}_d)(\boldsymbol{\sigma} - \mathbf{\Lambda}\tilde{\mathbf{q}}) \\ \dot{\boldsymbol{\sigma}} &= - \left[ \frac{\partial H}{\partial \mathbf{q}}(\mathbf{x}) + \mathbf{D} \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}) - \mathbf{G}(\mathbf{q})\mathbf{u} + \dot{\mathbf{p}}_r \right] \Big|_{\substack{\mathbf{q} = \tilde{\mathbf{q}} + \mathbf{q}_d \\ \mathbf{p} = \boldsymbol{\sigma} + \mathbf{p}_r}} \end{aligned} \quad (24)$$

The following result gives a solution to the control problem. For sake of space, some arguments are left out.

*Proposition 1.* Consider a twice differentiable desired trajectory  $\mathbf{x}_d \in T^*\mathcal{Q}$ , together with the change of coordinates (20) and (22). Consider also the pH system (14) in closed-loop with the control law<sup>2</sup>

$$\begin{aligned} \mathbf{G}\mathbf{u} &= \mathbf{G}\mathbf{u}_{eq} + \mathbf{G}\mathbf{u}_{at} \\ \mathbf{G}\mathbf{u}_{eq} &= \dot{\mathbf{p}}_r + \frac{\partial H}{\partial \mathbf{q}}(\mathbf{q}, \mathbf{p}_r) + \mathbf{D}(\mathbf{q}) \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}, \mathbf{p}_r) \\ \mathbf{G}\mathbf{u}_{at} &= -\mathbf{K}_d \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}, \boldsymbol{\sigma}) - \mathbf{M}^{-1}(\mathbf{q})\mathbf{\Lambda}\tilde{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}(\mathbf{p}_r^\top \mathbf{M}^{-1}(\mathbf{q})\boldsymbol{\sigma}), \end{aligned} \quad (25)$$

where  $\mathbf{K}_d$  fulfills

$$\mathbf{D} + \mathbf{K}_d + \frac{1}{2}\mathbf{I}_n - \frac{1}{4}(\mathbf{M}^{-1} + \mathbf{M}) > \mathbf{0}. \quad (26)$$

Then,

- (1) The closed-loop system in error coordinates (20) has a the form

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} -\mathbf{I} & \mathbf{I} \\ -\mathbf{I} & -(\mathbf{E}(\tilde{\mathbf{q}} + \mathbf{q}_d, \boldsymbol{\sigma}) + \mathbf{K}_d) \end{bmatrix} \begin{bmatrix} \mathbf{M}^{-1}\mathbf{\Lambda}\tilde{\mathbf{q}} \\ \mathbf{M}^{-1}\boldsymbol{\sigma} \end{bmatrix}. \quad (27)$$

- (2) The origin of (27) is exponentially stable with rate

$$\beta = \min \text{eig} \left( \mathbf{P}^{1/2}(\tilde{\mathbf{x}})\boldsymbol{\Upsilon}(\tilde{\mathbf{x}})\mathbf{P}^{1/2}(\tilde{\mathbf{x}}) \right). \quad (28)$$

where  $\min \text{eig}(\cdot)$  denotes the minimum eigenvalue of the matrix in the argument and matrices

$$\mathbf{P}(\tilde{\mathbf{x}}) = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1}(\tilde{\mathbf{q}} + \mathbf{q}_d) \end{bmatrix}, \quad (29)$$

$$\boldsymbol{\Upsilon}(\tilde{\mathbf{x}}) = \begin{bmatrix} 2\mathbf{M}^{-1} & (\mathbf{M}^{-1} - \mathbf{I}_n) \\ (\mathbf{M}^{-1} - \mathbf{I}_n) & 2(\mathbf{D} + \mathbf{K}_d) \end{bmatrix}. \quad (30)$$

- (3) The sliding manifold

$$\Omega(t) = \{\mathbf{x} \in T^*\mathcal{Q} : \boldsymbol{\sigma}(\mathbf{x}, t) = \tilde{\mathbf{p}}_\sigma + \mathbf{\Lambda}\tilde{\mathbf{q}} = \mathbf{0}\}, \quad (31)$$

where  $\tilde{\mathbf{p}}_\sigma := \mathbf{M}(\mathbf{q})\dot{\tilde{\mathbf{q}}}$ , is invariant and attractive, for system (27), with ideal sliding motion

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}, \mathbf{p}_r). \quad (32)$$

*Proof:*

- (1) Substitution of the control law (25) in the error system (24) and straightforward computations gives the closed-loop error system (27).

<sup>2</sup> With  $\frac{\partial}{\partial \mathbf{q}}(\mathbf{p}_r^\top \mathbf{M}^{-1}(\mathbf{q})\boldsymbol{\sigma}) = \mathbf{S}(\mathbf{q}, \boldsymbol{\sigma}) \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}, \mathbf{p}_r) + \mathbf{S}(\mathbf{q}, \mathbf{p}_r) \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}, \boldsymbol{\sigma})$ .

- (2) To prove this item, we will use partial contraction<sup>3</sup> Theorem 2. Motivated by (19), consider the following virtual system with state  $\tilde{\mathbf{x}}_a = [\tilde{\mathbf{q}}_a^\top, \boldsymbol{\sigma}_a^\top]^\top$

$$\dot{\tilde{\mathbf{x}}}_a = \begin{bmatrix} -\mathbf{I} & \mathbf{I} \\ -\mathbf{I} & -(\mathbf{E}(\tilde{\mathbf{q}} + \mathbf{q}_d, \boldsymbol{\sigma}) + \mathbf{K}_d) \end{bmatrix} \begin{bmatrix} \mathbf{M}^{-1}\mathbf{\Lambda}\tilde{\mathbf{q}}_a \\ \mathbf{M}^{-1}\boldsymbol{\sigma}_a \end{bmatrix}. \quad (33)$$

Notice  $\tilde{\mathbf{x}}_a = \tilde{\mathbf{x}}$  and  $\tilde{\mathbf{x}}_a = \mathbf{0}$  are two particular solutions of system (33). The variational dynamics of the virtual system (33) is

$$\delta \dot{\tilde{\mathbf{x}}}_a = - \begin{bmatrix} \mathbf{M}^{-1}\mathbf{\Lambda} & -\mathbf{M}^{-1} \\ \mathbf{M}^{-1}\mathbf{\Lambda} & (\mathbf{E} + \mathbf{K}_d)\mathbf{M}^{-1} \end{bmatrix} \delta \tilde{\mathbf{x}}_a. \quad (34)$$

For the prolonged system (33)-(34), let the candidate differential Lyapunov function be

$$V(\tilde{\mathbf{x}}_a, \delta \tilde{\mathbf{x}}_a, t) = \frac{1}{2} \delta \tilde{\mathbf{x}}_a^\top \mathbf{P}(\tilde{\mathbf{x}}) \delta \tilde{\mathbf{x}}_a. \quad (35)$$

The time derivative of (35) is

$$\begin{aligned} \dot{V} &= -\delta \tilde{\mathbf{x}}_a^\top \begin{bmatrix} \mathbf{\Lambda}\mathbf{M}^{-1}\mathbf{\Lambda} & -\mathbf{\Lambda}\mathbf{M}^{-1} \\ \mathbf{M}^{-2}\mathbf{\Lambda} & \mathbf{M}^{-1}(\mathbf{E} + \mathbf{K}_d)\mathbf{M}^{-1} \end{bmatrix} \delta \tilde{\mathbf{x}}_a \\ &\quad + \frac{1}{2} \delta \tilde{\mathbf{x}}_a^\top \begin{bmatrix} \dot{\mathbf{\Lambda}} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{M}}^{-1}(\mathbf{q}) \end{bmatrix} \delta \tilde{\mathbf{x}}_a, \\ &= -\delta \tilde{\mathbf{x}}_a^\top \begin{bmatrix} \mathbf{\Lambda}\mathbf{M}^{-1}\mathbf{\Lambda} & -\mathbf{\Lambda}\mathbf{M}^{-1} \\ \mathbf{M}^{-2}\mathbf{\Lambda} & \mathbf{M}^{-1}(\mathbf{D} + \mathbf{K}_d)\mathbf{M}^{-1} \end{bmatrix} \delta \tilde{\mathbf{x}}_a \\ &\quad + \delta \boldsymbol{\sigma}_a^\top \left[ -\mathbf{M}^{-1}(\mathbf{S} - \frac{1}{2}\dot{\mathbf{M}})\mathbf{M}^{-1} + \frac{1}{2}\dot{\mathbf{M}}^{-1} \right] \delta \boldsymbol{\sigma}_a, \\ &= -\delta \tilde{\mathbf{x}}_a^\top \underbrace{\begin{bmatrix} \mathbf{\Lambda}\mathbf{M}^{-1}\mathbf{\Lambda} & -\mathbf{\Lambda}\mathbf{M}^{-1} \\ \mathbf{M}^{-2}\mathbf{\Lambda} & \mathbf{M}^{-1}(\mathbf{D} + \mathbf{K}_d)\mathbf{M}^{-1} \end{bmatrix}}_{\boldsymbol{\Xi}(\tilde{\mathbf{x}})} \delta \tilde{\mathbf{x}}_a, \end{aligned} \quad (36)$$

where the symmetric part of  $\boldsymbol{\Xi}(\tilde{\mathbf{x}})$  is expressed as

$$\text{Sym}(\boldsymbol{\Xi}(\tilde{\mathbf{x}})) = \frac{1}{2} \mathbf{P}(\tilde{\mathbf{x}}) \boldsymbol{\Upsilon}(\tilde{\mathbf{x}}) \mathbf{P}(\tilde{\mathbf{x}}) \quad (37)$$

Thus, (36) will be negative definite if and only if the Schur complement of matrix (30) with respect to  $2\mathbf{M}^{-1}$  fulfills (26). Which is always possible by choosing a big enough  $\mathbf{K}_d$ . Therefore, the prolonged system (33)-(34) contracts (35) with respect to the metric (29) in  $T\mathcal{X}$ .

With (28), the time derivative (36) satisfies

$$\dot{V}(\tilde{\mathbf{x}}_a, \delta \tilde{\mathbf{x}}_a, t) < -2\beta V(\tilde{\mathbf{x}}_a, \delta \tilde{\mathbf{x}}_a, t) \quad (38)$$

uniformly in  $t$ , or equivalently (8) for the matrix (29). By Remark 1, the virtual system (33) is incrementally exponentially stable with rate (28). Therefore,  $\tilde{\mathbf{x}}$  converges to  $\mathbf{0}$  exponentially with rate  $\beta$ , as  $t \rightarrow \infty$ .

- (3) The existence of  $\mathbf{u}_{eq}$  in (25), guarantees that the sliding manifold (31) is rendered to an invariant manifold. From the previous item,  $\boldsymbol{\sigma} \rightarrow \mathbf{0}$ , which means the invariant manifold is attractive. Finally, system (14) has *regular canonical form*, and definitions of  $\mathbf{p}_r$  and  $\boldsymbol{\sigma}$  imply, by straightforward computations, the *reduced-order* ideal sliding motion

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}, \mathbf{p}_r). \quad (39)$$

<sup>3</sup> By defining  $\mathbf{e} = \tilde{\mathbf{x}}_a$  and  $\mathbf{x} = \tilde{\mathbf{x}}$ , properties labeled as TULES-NL, UES-TL and ULMTE in Andrieu et al. (2016), are also verified.

In Sanfelice and Praly (2015), an observer was designed for shrinking a Riemannian distance, instead of designing a contracting observer. Our controller has the same function as the observer in that paper.

*Proposition 2.* Consider system (14). The control law (25) shrinks the Riemannian distance  $d(\mathbf{x}, \mathbf{x}_d)$  induced by

$$\mathbf{\Pi}(\mathbf{x}) = \begin{bmatrix} \mathbf{\Lambda} + \mathbf{\Lambda} \mathbf{M}^{-1}(\mathbf{q}) \mathbf{\Lambda} & \mathbf{\Lambda} \mathbf{M}^{-1}(\mathbf{q}) \\ \mathbf{M}^{-1}(\mathbf{q}) \mathbf{\Lambda} & \mathbf{M}^{-1}(\mathbf{q}) \end{bmatrix}. \quad (40)$$

*Proof:* We will show system (14) is partially contracting to a virtual system with respect to the metric (40), in the sense of Remark 3, such that the induced distance by the contraction metric converges exponentially to zero.

The controller (25) can be expressed in implicit form by

$$\begin{aligned} \dot{\mathbf{x}}_{d\sigma} = & \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & -\mathbf{D} - \mathbf{S}(\mathbf{q}, \tilde{\mathbf{p}}_\sigma) \end{bmatrix} \frac{\partial H}{\partial \mathbf{x}}(\mathbf{q}, \mathbf{p}_{d\sigma}) + \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix} \mathbf{u} \\ & + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{A}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}_\sigma, t) & \mathbf{B}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}_\sigma, t) \end{bmatrix} \frac{\partial \tilde{H}}{\partial \tilde{\mathbf{x}}}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}_\sigma, t), \end{aligned} \quad (41)$$

where  $\mathbf{x}_{d\sigma} = [\mathbf{q}_d, \mathbf{p}_{d\sigma}]^\top$  is the desired trajectory and

$$\begin{aligned} \mathbf{A} &:= -\frac{1}{2} \dot{\mathbf{M}} + \mathbf{D} + \mathbf{S}(\mathbf{q}, \tilde{\mathbf{p}}_\sigma + \mathbf{\Lambda} \tilde{\mathbf{q}}) + \mathbf{K}_d + \mathbf{I}, \\ \mathbf{B} &:= \mathbf{K}_d + \mathbf{\Lambda} - \mathbf{S}(\mathbf{q}, \mathbf{p}_{d\sigma} - \mathbf{\Lambda} \tilde{\mathbf{q}}). \end{aligned} \quad (42)$$

Let a virtual system with state  $\mathbf{x}_v = [\mathbf{q}_v, \mathbf{p}_v]^\top$  be

$$\begin{aligned} \dot{\mathbf{x}}_v = & \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & -\mathbf{E}(\mathbf{q}, \mathbf{p}_{d\sigma} + \tilde{\mathbf{p}}_\sigma) \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial \mathbf{q}}(\mathbf{q}) \\ \frac{\partial \tilde{H}}{\partial \mathbf{p}}(\mathbf{q}, \mathbf{p}_v) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix} \mathbf{u} \\ & + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{A}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}_\sigma, t) & \mathbf{B}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}_\sigma, t) \end{bmatrix} \frac{\partial \tilde{H}}{\partial \tilde{\mathbf{x}}}(\tilde{\mathbf{x}}_v, t), \end{aligned} \quad (43)$$

with  $\tilde{\mathbf{x}}_v = \mathbf{x} - \mathbf{x}_v$ . Notice system (43) has as particular solutions to both,  $\mathbf{x}_v = \mathbf{x}_{d\sigma}$  and  $\mathbf{x}_v = \mathbf{x}$ . The variational dynamics of (43) is

$$\delta \dot{\mathbf{x}}_v = - \begin{bmatrix} \mathbf{0} & -\mathbf{M}^{-1} \\ \mathbf{A} \mathbf{M}^{-1} \mathbf{\Lambda} & (\mathbf{E} + \mathbf{B}) \mathbf{M}^{-1} \end{bmatrix} \delta \mathbf{x}_v. \quad (44)$$

Now, consider the following change of coordinates

$$\delta \tilde{\mathbf{x}}_a = - \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{\Lambda} & \mathbf{I} \end{bmatrix} \delta \mathbf{x}_v = -\mathbf{\Theta} \delta \mathbf{x}_v. \quad (45)$$

Then, system (44) in the new coordinates is

$$\delta \dot{\tilde{\mathbf{x}}}_a = -\mathbf{\Theta} \begin{bmatrix} \mathbf{0} & -\mathbf{M}^{-1} \\ \mathbf{A} \mathbf{M}^{-1} \mathbf{\Lambda} & (\mathbf{E} + \mathbf{B}) \mathbf{M}^{-1} \end{bmatrix} \mathbf{\Theta}^{-1} \delta \tilde{\mathbf{x}}_a \quad (46)$$

which is nothing but (34). Then, the virtual system (43) is contracting with rate  $2\beta$ , with respect to the differential Lyapunov function or contraction measure

$$\bar{V}(\mathbf{x}_v, \delta \mathbf{x}_v) = \frac{1}{2} \delta \mathbf{x}_v^\top \mathbf{\Pi}(\mathbf{q}, \mathbf{p}) \delta \mathbf{x}_v \quad (47)$$

where  $\mathbf{\Pi}(\mathbf{x}) = \mathbf{\Theta}^\top \mathbf{P}(\tilde{\mathbf{x}}) \mathbf{\Theta}$ . Thus,  $\bar{V}(\mathbf{x}_v, \delta \mathbf{x}_v) < e^{-2\beta t}$ .

Now, as in Forni and Sepulchre (2014), consider the set  $\Gamma(\mathbf{x}, \mathbf{x}_{d\sigma})$  of all normalized paths  $\gamma : [0, 1] \rightarrow \mathcal{X}$  connecting  $\mathbf{x}$  with  $\mathbf{x}_{d\sigma}$  such that  $\gamma(0) = \mathbf{x}$  and  $\gamma(1) = \mathbf{x}_{d\sigma}$ . Function (47) defines a *Finsler* structure in  $T\mathcal{X}$ , which by integration induces the distance

$$d(\mathbf{x}, \mathbf{x}_{d\sigma}) = \inf_{\Gamma(\mathbf{x}, \mathbf{x}_{d\sigma})} \int_0^1 \sqrt{\bar{V}(\gamma(s), \frac{\partial \gamma}{\partial s}(s))} ds < e^{-\beta t}. \quad (48)$$

Therefore,  $d(\mathbf{x}, \mathbf{x}_{d\sigma}) \rightarrow 0$  as  $t \rightarrow \infty$ . ■

#### 4. CASE OF STUDY: 3 DOF SCARA ROBOT

Consider a SCARA robot with configuration manifold  $\mathcal{Q} = S^1 \times S^1 \times \mathbb{R}$ , and  $S^1$  the unitary circumference. The generalized position vector  $\mathbf{q}^\top = [\theta_1, \theta_2, z]$ , generalized momentum  $\mathbf{p}^\top = [p_{\theta_1}, p_{\theta_2}, p_z]$  and generalized force  $\mathbf{u}^\top = [\tau_1, \tau_2, f]$ . Such system has a pH representation of form (14), where the inertia matrix is given by

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{12} & m_3 l_2^2 & 0 \\ 0 & 0 & (m_1 + m_2 + m_3)g \end{bmatrix}, \quad (49)$$

and

$$\begin{aligned} M_{11} &= (m_2 + m_3)l_1^2 + m_3 l_2^2 + 2m_3 l_1 l_2 \cos \theta_2, \\ M_{12} &= m_3 l_2^2 + m_3 l_1 l_2 \cos \theta_2. \end{aligned}$$

The potential energy is  $V(\mathbf{q}) = (m_1 + m_2 + m_3)gz$  and input matrix  $\mathbf{G}(\mathbf{q}) = \mathbf{I}_3$ .

The goal is to track to  $\mathbf{q}_d = [\sin(t) + 1, \sin(t), \sin(t)]^\top$ , by closing the loop with the control scheme (25), with gain matrices  $\mathbf{\Lambda} = \text{diag}\{15, 15, 15\}$  and  $\mathbf{K}_d = \text{diag}\{30, 60, 90\}$ .

In Figure 1, the time responses of the error variables are shown. All converge to zero exponentially after transients too. Notice the zero steady-state value of time response of  $\tilde{\mathbf{q}}$  is guaranteed by  $\sigma = \mathbf{0}$ . Above is actually the reason due to  $\tilde{\mathbf{q}}$  and  $\tilde{\mathbf{p}}$  converge slower than the sliding variable.

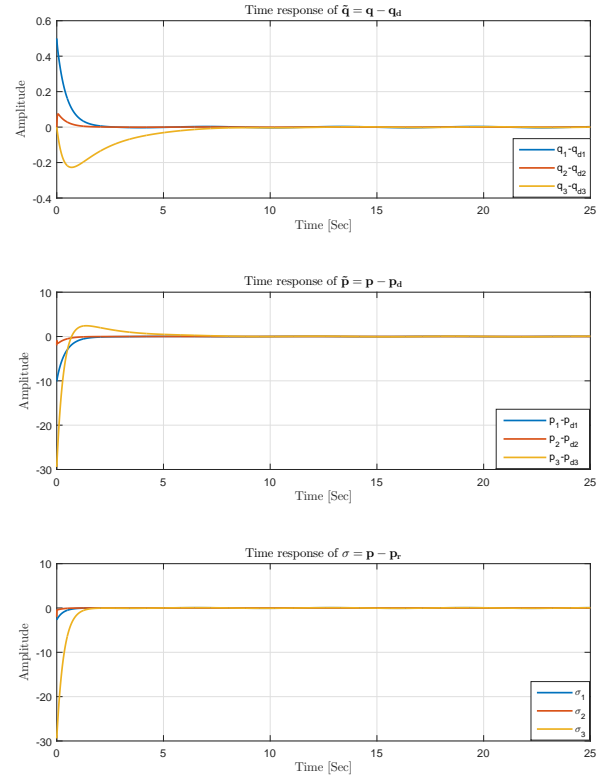


Fig. 1. Position and momentum error and sliding variable

Upper plot of Figure 2 shows the time response of the contraction measure with respect to the desired trajectory (assuming  $\gamma$  is a straight line), which after an overshoot transient, in fact shrinks. This is reflected in the lower plot shows the time response of the Hamiltonian versus desired Hamiltonian function.

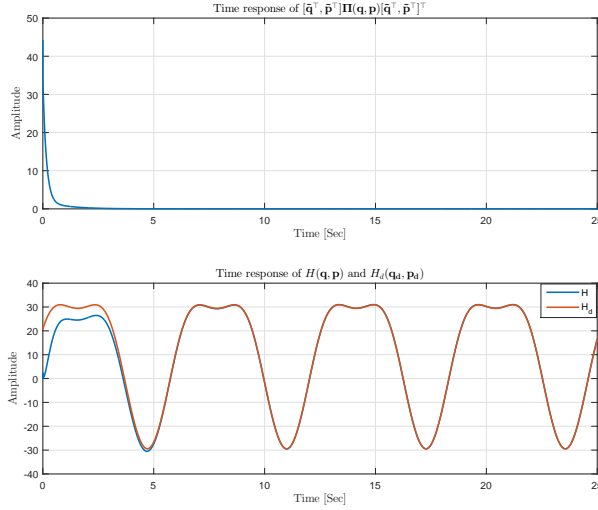


Fig. 2. Contraction measure and Hamiltonian functions

The control effort is shown in Figure 3. It converges to a steady state trajectory smoothly, after a big transient. This due to the (18) has a big transient, and the controller was required to compensate it.

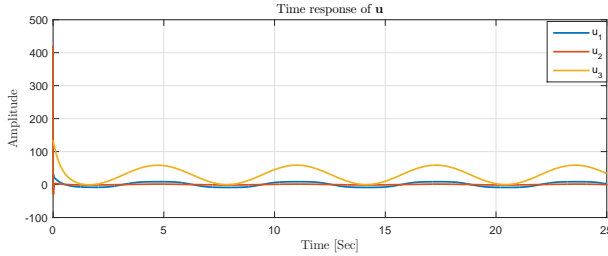


Fig. 3. Control signal time response

## 5. CONCLUSIONS

In this paper we presented a trajectory tracking control design method for fully-actuated port-Hamiltonian systems by rendering a sliding manifold to an attractive invariant manifold. The control law is composed by the equivalent control, which renders the sliding surface to an invariant set; and attractivity was done by giving a partially contracting pH structure to the closed-loop error system. Moreover, the controller contracts exponentially a Riemannian distance as result of incremental stability properties of the virtual system. Simulations showed the good performance of the proposed controller.

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